ARTICLE NO. 80-4019

Missile Guidance Laws Based on Singular Perturbation Methodology

B. Sridhar* and N.K. Gupta†
Systems Control, Inc. (Vt), Palo Alto, Calif.

Air-to-air missile guidance laws pose significant design issues of effectiveness, complexity, and real-time computation requirements, particularly when they are designed for use against maneuvering targets. Conventional homing and proportional navigation (pro nav) guidance laws are simple and require minimal computation. These approaches are, however, not generally effective against maneuvering targets, leading to launch envelopes of reduced size. Guidance laws based on optimal control theory may minimize the miss distance, but the resulting optimization problem is complex and difficult to implement in real time. The singular perturbation approach described in this paper provides 1) an approximate method to solve the optimal problem and 2) a basis to relate this solution to conventional and pro nav control laws. Physical properties of missiles, targets, and engagement scenarios are explicitly considered in this paper to maximize the accuracy of a guidance law based on singular perturbation methodology.

Nomenclature

a_t = magnitude of target acceleration
a_m = magnitude of missile acceleration
$a_{\max}$ = maximum value of a_m
C_{d0} = zero-lift drag coefficient
D = drag
$D^{[\cdot]}$ = drag corresponding to $[\cdot]$ layer
g = gravity
g_m = load factor limit
H = Hamiltonian
$H^{[\cdot]}$ = Hamiltonian corresponding to $[\cdot]$ layer
J = performance index
k = target turn rate
K = induced drag coefficient
K_{nav} = navigation gain
l = estimate of the distance traveled by the missile
L = lift
$L_{\max}$ = maximum value or lift
$\tilde{L}$ = lift in analysis axis, $\tilde{L}^0$ lift in analysis or outer solution
L_2,L_3 = components of lift in x-y and y-z plane, respectively
m = mass of the missile
m_0 = initial mass of the missile
q = dynamic pressure
q_0 = initial value of dynamic pressure
s = reference area of the missile
t = time
t_f = final time
T = thrust
v = speed of the missile
v_c = closing velocity of the missile
v_0 = initial speed of the missile
$\bar{v}_d$ = desired direction of missile velocity
v_m = missile velocity
v_t = target speed
v_1,v_2 = components of virtual target velocity
w = missile weight
x,y,z = relative position of the missile from the target
$x(0),y(0)$ = initial value of x and y
x_r,y_r = relative position of the missile in the analysis axes
δ = orientation of missile acceleration
γ = missile flight path angle
$\gamma^{[\cdot]}$ = missile flight path angle in $[\cdot]$ layer
ϕ = missile heading in first analysis axes
ϕ_t = target heading in first analysis axes
ρ = air density
$\lambda^{[\cdot]}_{(\cdot)}$ = adjoint variable corresponding to state $(\cdot)$ in $[\cdot]$ layer
ϵ = time-stretching parameter
ξ = heading in second analysis axes
$\phi^o,\gamma^o,\epsilon^o$ = optimum values of ϕ,γ,ξ in the outer solution
σ = line of sight angle
τ = stretched time
$\mathcal{L}$ = integrand in the performance index

Superscripts

o = outer layer
1 = boundary layer
$(\dot{})$ = differentiation with respect to time

Introduction

HIGHLY maneuvering targets of the next decade pose a serious threat to current tactical missiles. Three major areas where tactical missiles need improvement are propulsion, seekers, and guidance and control. In the end game, missile aerodynamic capabilities are often not fully utilized because of poor guidance logic. Therefore, advanced guidance laws provide an effective means to combat high-g targets. Recent developments in digital hardware enable implementation of such guidance algorithms with small additional cost.

Early missiles used a pursuit form of navigation in which steering commands are generated to drive the look angle to zero. The missile heads in the direction of current target position. The control strategy is satisfactory for stationary targets and leads to tail chases for moving targets. An extension of this approach is called proportional navigation or pro nav, and works well for constant velocity targets. In pro nav (PN) the line-of-sight rate is driven to zero by lateral acceleration commands proportional to line-of-sight rates. A

Received Dec. 11, 1978; revision received July 20, 1979.

Index categories: Guidance and Control; Missile Systems; LV/M Dynamics and Control.

*Senior Engineer, Advanced Technology Group.
†Manager, Advanced Technology Group.

simple extension of previous control laws may be used for targets with nonzero longitudinal acceleration but no lateral acceleration. The pro nav and its various extensions form the basis of guidance laws used in all tactical air-to-air missiles today.[1-5]

The missile-target engagement problem may be formulated as an optimal control with a specified performance function. The resulting solution, however, is so complex that it cannot be implemented on microprocessors likely to be available during the next ten years. The singular perturbation (SP) method presented in this paper leads to a simplified solution of the optimal control problem.

A general theory has been developed for multiple time scale problems in linear systems. No such methodology is available for nonlinear systems. The success of the technique with nonlinear dynamics depends primarily on problem formulation. A critical variable is the set of time scales whose ratio is defined to be small. In our approach, this small parameter is introduced based on the physical properties of missiles, targets, and typical engagement scenarios. The value of this parameter for a particular case indicates the accuracy of the approximate solution and also determines areas where higher order approximations to the solution are necessary. The resulting solution is nearly optimal. Different formulations will, however, be required for different missiles.

The missile/target characteristics are often such that nonuniform approximations are required in different systems of equations. A solution based on a particular set of nonuniform approximations is presented in this paper. The control law is further simplified by 1) defining several axis systems, one for each set of fast and slow equations; 2) deriving an equivalent nonmaneuvering target; and 3) utilizing the feedback nature of the implementation. The resulting control law may be implemented using current digital hardware. Its performance is significantly better than pro nav against highly maneuvering targets, particularly in difficult engagement scenarios.

The next section gives the problem formulation followed by the discussion of general singular perturbation methods. The rationale behind the selection of slow and fast states is described and this selection is used to specify singular perturbation guidance for missiles. The following section considers the implementation of the SP guidance law and compares it with PN guidance. An example is given followed by conclusions and general discussion of the guidance law.

Problem Formulation

A typical engagement scenario is assumed in which the missile and the target start from arbitrary points in space with general initial velocities. The target maneuvers with a constant acceleration a_t perpendicular to its velocity.‡ Since both the missile and the target acceleration capabilities are much higher than 1 g, the direction of gravity is not important and any convenient axis system may be selected. The following system of axes is selected initially to simplify the derivation of the control laws: 1) the x-axis is chosen to be along the initial direction of the target velocity vector; 2) the y-axis is perpendicular to the x-axis in the plane of rotation of the target; 3) the z-axis is perpendicular to the plane of rotation. We will refer to this set of axes as the first analysis axes (FAA).

Missile and Target Dynamics

The equations of motion describing the intercept problem in the FAA, ignoring gravity, are

$$\dot{x} = v \cos\gamma \cos\phi - v_t \cos\phi_t \tag{1}$$

$$\dot{y} = v \cos\gamma \sin\phi - v_t \sin\phi_t \tag{2}$$

‡Targets often perform random lateral accelerations. This kind of evasive maneuver may be used for guidance law computation. Computation with a six-degree-of-freedom simulation has shown no performance degradation when targets use other evasive maneuvers.

$$\dot{z} = v \sin\gamma \tag{3}$$

$$\dot{v} = (T-D)/m \tag{4}$$

$$\dot{\phi}_t = a_t/v_t \tag{5}$$

$$\dot{\phi} = a_m \sin\delta / v \cos\gamma \tag{6}$$

$$\dot{\gamma} = a_m \cos\delta / v \tag{7}$$

The various angles are illustrated in Fig. 1.

Drag is a nonlinear function of velocity. For the purpose of control design, it is modeled by the parabolic drag form

$$D = qsc_{d0} + KL^2/qs \tag{8}$$

Further,

$$L = m \cdot a_m \qquad L_2 = L \sin\delta \qquad L_3 = L \cos\delta \qquad q = \tfrac{1}{2}\rho v^2 \tag{9}$$

The thrust is T for $t \le t_T$ and is zero for $t > t_T$. The control in this problem is the lift L. Lift is chosen subject to the constraint

$$L \le w g_m(v) \tag{10}$$

where $g_m(v)$ represents a load factor limit which may arise due to a structural limit, control surface actuator limit, or autopilot stability considerations. It is, in general, a function of missile speed.

Control Requirements

The missile intercept guidance problem involves selecting the lift vector to drive the missile from its present state to the same position as the target

$$x(t_f) = y(t_f) = z(t_f) = 0 \tag{11}$$

and the final time t_f is free. The constraints are that the lift vector should lie in a plane perpendicular to missile velocity and its magnitude should be smaller than the maximum allowable lift. A general performance index is specified

$$J = \int_0^{t_f} \mathcal{L} \, dt \tag{12}$$

where $\mathcal{L}$ is some function of missile states and controls (other than position and orientation). Final values of selected state

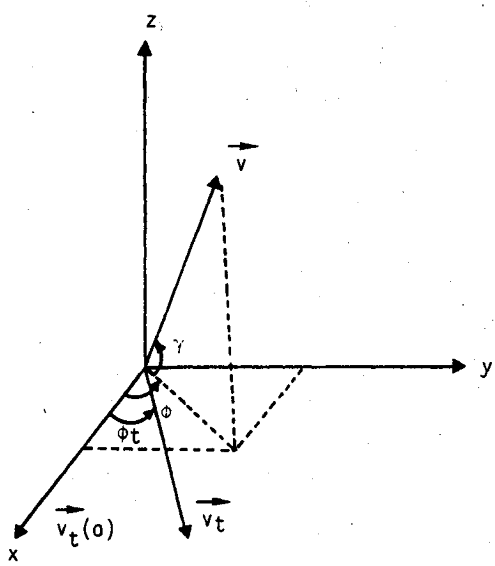

Fig. 1 First analysis axes (FAA) and angles.

variables may also be included in the cost functional if desired by using a Dirac delta function term in $\mathcal{L}$.

Optimal Solution

The Lagrange multiplier method may be used to find the optimal solution. The Hamiltonian for this problem is

$$H=\lambda_x(v\cos\gamma\cos\phi-v_t\cos\phi_t)+\lambda_y(v\cos\gamma\sin\phi-v_t\sin\phi_t)$$
$$+\lambda_z v\sin\gamma+\lambda_v(T-D)/m+\lambda_{\phi_t}\cdot a_t/v_t$$
$$+\lambda_\phi\cdot gL_2/wv\cos\gamma+\lambda_\gamma gL_3/wv+\mathcal{L} \tag{13}$$

Since the Hamiltonian does not depend on x, y or z,

$$\dot\lambda_x=\dot\lambda_y=\dot\lambda_z=0 \tag{14}$$

The other four Lagrange variables are

$$\dot\lambda_v = -\partial H/\partial v \tag{15}$$

$$\dot\lambda_\phi = -\partial H/\partial \phi \tag{16}$$

$$\dot\lambda_\gamma = -\partial H/\partial \gamma \tag{17}$$

$$\dot\lambda_{\phi_t} = -\partial H/\partial \phi_t \tag{18}$$

The final value of all these adjoint variables is zero (no final condition constraints on v, ϕ, γ, and ϕ_t).

The solution for the optimal control L and δ and the final time, t_f, results in a two-point boundary value problem (TPBVP) involving Eqs. (1) through (18). Several numerical methods[6] are available for the solution of the TPBVP and all of them result in excessive computation and open-loop solutions. A suboptimal solution to the TPBVP based on singular perturbation theory needs to be developed.

Summary of Singular Perturbation Methodology

In the singular perturbation methodology, the state vector is divided into a slower part and a faster part.[7] Qualitatively, the slower part has a larger time constant than the faster part. The state equations are written by introducing an arbitrary small parameter ϵ:

$$\dot x = f(x,y,u) \qquad x(0)=x_0 \tag{19a}$$

$$\epsilon\dot y = g(x,y,u) \qquad y(0)=y_0 \tag{19b}$$

The optimal control problem of minimizing a general cost functional

$$J=\int_0^{t_f}\mathcal{L}(x,y,u)\mathrm{d}t \tag{20}$$

is reduced to the solution of two subproblems.

Step 1: Set $\epsilon\to 0$. This reduces the differential equation [Eq. (19b)] to an algebraic equation. Solve Eq. (19b) for $y=y_s$ in terms of x and u. Substitute for y in Eqs. (19a) and (20). Solve the optimization problem to compute $u^*(0)=u_s$ and solve for x_s^* from Eq. (19a). Physically this step means that the variation of y is so fast compared to x that y can be assumed to have reached a steady state while computing u_s and x_s^*.

Step 2: This step considers the effect of the simplification on the fast variable y. Because of the simplification, there is a discrepancy between y_s and y_0. Thus, we are looking for an approximation

$$y(t)=y_s+O(\epsilon) \tag{21}$$

that is valid for $t>0$. To study the behavior of y, consider the time transformation

$$\tau=t/\epsilon \tag{22}$$

In the stretched time, Eqs. (19a) and (19b) can be written as

$$\frac{\mathrm{d}x}{\mathrm{d}\tau}=\epsilon f[x(\tau),y(\tau),u(\tau)] \tag{23a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau}=g[x(\tau),y(\tau)u(\tau)] \tag{23b}$$

From Eq. (23b), even as $\epsilon\to 0$ and $\dot y\to\infty$, $\epsilon\dot y$ is finite, i.e., the rate of change of the fast variable in the stretched time scale is finite. Further, in the stretched time scale x remains at its initial value x_0. Also $u=u^*(0)=u_s$. Thus the behavior of the fast variable as a function of τ can be represented by

$$\frac{\mathrm{d}y}{\mathrm{d}\tau}=g[x(0),y(\tau),u_s] \qquad y(0)=y_0 \tag{24}$$

The fast solution is improved by introducing a fast control u_f into the Eq. (24),

$$\frac{\mathrm{d}y}{\mathrm{d}\tau}=g[x(0),y(\tau),u_f(\tau)] \tag{25}$$

and require that

$$\lim_{\tau\to\infty} u_f(\tau)=u_s \tag{26}$$

Equations (25) and (20) are used to compute the fast control. The optimal control u^* is approximated by

$$u_0^*=u_s(t)+u_f(\tau)-u_s+O(u) \tag{27}$$

Solve Eq. (25) for y_0^*.

Step 3: In step 2, it was assumed that the slow variables and the control remain at their initial values x_0 and $u^*(0)$, respectively. Corrections can be applied whenever this assumption is not valid. Substitute the more accurate values of y_0^* and u_0^* in Eq. (19a) and solve the first problem again to get an ϵ-order correction. This procedure could be repeated with Eq. (19b) to obtain second-order correction. Every repetition of the procedure gives a higher-order correction term. A nonuniform solution applies different order correction to the slow and the fast system.

The main decision in the application of the SP methodology is the classification of states into faster and slower parts. This should be based on the physical characteristics of the system and the performance index. The next decision involves the selection of a proper combination of faster states and controls while solving the reduced-order optimization problems in steps 1 and 2.

Selection of Faster and Slower States

Physically, the dynamics of the interceptor problem may be divided into seven parts: 1) relative position (x,y,z), 2) missile speed (v), 3) missile velocity orientation (ϕ,γ), 4) missile acceleration or missile pitch dynamics (L), 5) orientation of missile acceleration or roll dynamics (δ), 6) target velocity, and 7) target acceleration. In the computation of the missile guidance laws, the future time histories of the target velocity and acceleration are not known and must be set to some values (for example, by means of differential games analysis, or from observed target behavior). Therefore, the last two sets of variables do not enter the dynamic states considered here.

Table 1 Relative dynamics of missile and target states

Vehicle	Dynamics	States	Characteristic value	Characteristic time, s
Missile	Position	x,y,z	2000 ft	1.0-2.0
	Speed	v	1000 ft/s	10.0-15.0
	Velocity orientation	γ,ϕ	30 deg	0.2-0.3
	Acceleration (autopilot pitch dynamics)	L	max value	~0.1
	Acceleration direction (autopilot roll dynamics)	δ	±90 deg	~0.01
Target	Velocity	v_t	1000 ft/s	3.0-15.0
	Acceleration	a_t	320 ft/s^2	0.2-0.5

Table 1 gives a typical set of time constants for each of the seven parts of the missile/target dynamics. The time constants are based on the variations of the states about the nominal or zero input values. The missile speed, for example, varies quite rapidly, but the additional drag due to the control input produces a deceleration of about 60-100 ft/s^2. Based on the time constants, the states are classified into four groups: 1) velocity (very slow), 2) position (slow), 3) velocity orientation (fast), and 4) acceleration and its direction (very fast).

Since the slow system has a time constant in the unit range, we can define small parameters of the SP method based on the time constants of the particular equations under consideration. Essentially, we scale the very slow, fast, and very fast equations such that the variation of the scaled variables is of the same order of magnitude as the slow variable. By introducing these parameters, the above system of equations may be written as

Very slow

$$\frac{1}{\mu}\left(\frac{\dot{v}}{v_0}-\frac{T-qsC_{d0}}{mv_0}\right)=-\frac{v_0}{d}\cdot\left(\frac{L}{L_{\max}}\right)^2\frac{m_0q_0}{mq} \tag{28}$$

$$\mu=\frac{KL^2_{\max}}{q_0sm_0}\frac{d}{v_0^2} \tag{29}$$

d = characteristic distance

Slow

$$\frac{\dot{x}}{d}=\frac{v}{d}\cos\gamma\,\cos\phi-\frac{v_t}{d}\cos\phi_t,\ \text{etc.} \tag{30}$$

Fast

$$\epsilon\dot{\phi}=\frac{v_0}{v}\cdot\frac{v_0}{d}\cdot\frac{L}{L_{\max}}\frac{\sin\delta}{\cos\gamma} \tag{31}$$

$$\epsilon\dot{\gamma}=\frac{v_0}{v}\cdot\frac{v_0}{d}\cdot\frac{L}{L_{\max}}\cos\delta \tag{32}$$

$$\epsilon=v_0^2/d a\max \tag{33}$$

Very fast

$$\epsilon_1\dot{L}=(v/d)(L_C-L) \tag{34}$$

$$\epsilon_2\dot{\delta}=(v/d)(\delta_C-\delta) \tag{35}$$

L_C and δ_C are the commanded variables. ϵ_1 and ϵ_2 are the time constants of pitch and roll autopilot dynamics scaled with d/v.

Singular-Perturbation-Based Intercept Guidance Laws

It was pointed out in the previous section that the states of the interceptor problem can be arranged in a hierarchy depending on the speed of response. In a short-range air-to-air missile, there is no control on engine thrust. Therefore, the velocity of the missile is a function of time and not subject to optimization. For simplicity, in the subsequent analysis the velocity (very slow) and position (slow) states are considered slow states. Also, in the present analysis we ignore the pitch and roll dynamics. Thus the system is divided into slow and fast dynamics for the purpose of deriving singular-perturbation-based guidance laws. The faster subsystem should be modified to

$$\epsilon\dot{\phi}=\frac{v_0}{v}\cdot\frac{v_0}{d}\cdot\frac{L}{L_{\max}}\cdot\frac{\sin\delta}{\cos\gamma} \tag{36}$$

$$\epsilon\dot{\gamma}=\frac{v_0}{v}\cdot\frac{v_0}{d}\frac{L}{L_{\max}}\cos\delta \tag{37}$$

The transformed state equations (36) and (37) modify the adjoint equations (16) and (17) to

$$\epsilon\dot{\lambda}_\phi=-\partial H/\partial\phi \tag{38}$$

$$\epsilon\dot{\lambda}_\gamma=-\partial H/\partial\gamma \qquad \lambda_\gamma(t_f)\ \text{free} \tag{39}$$

The accuracy of the singular perturbation approximation increases as ϵ decreases since approximations are sought by expanding the solution around $\epsilon=0$. The characteristic distance d can be considered to be the initial range between the missile and the target. Table 2 shows the value of ϵ for different ranges for a 30 g and a 100 g missile. The initial speed of the missile is 1000 ft/s. The approximation, therefore, improves as the missle/target separation and the missile acceleraiton capability are increased or the missile speed is decreased. The actual value of ϵ does not play any role if only step 1 and step 2 are used in the computation of the guidance law. However, a knowledge of ϵ is required to decide whether higher-order corrections (step 3) are necessary and to carry out these corrections. A study of Table 2 shows that the singular perturbation method provides an excellent basis for computing control laws in bank-to-turn missiles.

Table 2 Small parameter ϵ as a function of initial range and maximum missile acceleration

Maximum missile acceleration	Range, ft 2000	5000	10,000
30 g	0.520	0.208	0.104
100 g	0.155	0.062	0.031

Slow or Large Time Constant Solution (Outer Solution)

In the slow solution we let $\epsilon \to 0$. Then x, y, z are the state variables and ϕ and γ are the control variables. The intercept condition can be shown[8] to give

$$x(0) + l\cos\gamma^o\cos\phi^o - v_t/k\sin(kt_f) = 0 \tag{40}$$

$$y(0) + l\cos\gamma^o\sin\phi^o + v_t/k[\cos(kt_f) - 1] = 0 \tag{41}$$

$$z(0) + l\sin\gamma^o = 0 \tag{42}$$

where $k = a_t/v_t$. Since t_f is free and the Hamiltonian is not an explicit function of time,

$$H^o(t) = 0 \tag{43}$$

on the optimal trajectory. In particular,

$$H^o(t_f) = 0 \tag{44}$$

The intercept condition can be used to solve for t_f, l, γ^o and ϕ^o. The adjoint variables λ_x^o, λ_y^o and s_z^o are constants. $\lambda_{\phi_t}^o$ and λ_v^o are

$$\lambda_{\phi_t}(t) = v_t/k[\lambda_x^o(\cos kt - \cos kt_f) + \lambda_y^o(\sin kt - \sin kt_f)] \tag{45}$$

$$\lambda_v^o = -\frac{m}{(T-D)}[\lambda_x^o(v\cos\gamma^o\cos\phi^o - v_t\cos kt)$$
$$s_y^o(v\cos\gamma^o\sin\phi^o - v_t\sin kt) + \lambda_{\phi_t}^o\cdot k + \mathcal{L}] \tag{46}$$

An important contribution of this paper is the simplification of the solution by showing that the missile-intercept problem for the maneuvering target can be reduced to the intercept of an equivalent constant velocity target. By eliminating λ_{ϕ_t}, the Hamiltonian for the slow part of the system becomes

$$H^o = \lambda_x^o(v\cos\gamma\cos\phi - v_t\cos kt_f) + \lambda_y^o(v\cos\gamma\sin\phi - v_t\sin kt_f)$$
$$+ \lambda_z^o v\sin\gamma + \lambda_v^o(T-D)/m + \mathcal{L}^o = 0 \tag{47}$$

$$\lambda^o{}_v = -[m/(T-D)\lambda_x^o(v\cos\gamma^o\cos\phi^o - v_t\cos kt_f)$$
$$+ \lambda_y^o(v\cos\gamma^o\sin\phi^o - v_t\sin kt_f) + \mathcal{L}^o] \tag{48}$$

An examination of Eq. (13), (47), and (48) reveals that the effect of a maneuvering target in the computation of the adjoints λ_x^o, λ_y^o, λ_z^o and λ_v^o is equivalent to that of a virtual target which flies at constant velocity, $[v_t\cos kt_f, v_t\sin kt_f, 0]$.

Let ξ be the angle between $\vec{v}_d$ and the missile velocity vector $\vec{v}$. The key to the solution simplification is the definition of a new set of axes with 1) x-axis along v_d, 2) z-axis being the normal to plane containing $\vec{v}_m$ and $\vec{v}_d$, and 3) y-axis perpendicular to the x-axis. This set of axes is referred to as the second analysis axes (SAA). Let v_1 and v_2 be components of the virtual target velocity along the SAA x- and y-axes.

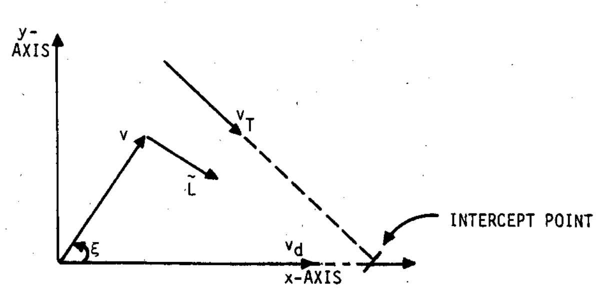

Fig. 2 Forces in the SAA.

The equations of motion in the SAA are

$$\dot{x}_r = v\cos\xi - v_1 \tag{49}$$

$$\dot{y}_r = v\sin\xi - v_2 \tag{50}$$

$$\dot{v} = (T-D)/m \tag{51}$$

$$\epsilon\dot{\xi} = v_o^2\tilde{L}/vdL_{\max} \tag{52}$$

where x_r and y_r are the relative distances of the missile from the virtual target along the SAA x- and y-axes. $\tilde{L}$ is the lift force acting in the $(v - v_d)$ plane as shown in Fig. 2. The outer solution Hamiltonian in the SAA is

$$H^{or} = \lambda_{x_r}(v\cos\xi - v_1) + \lambda_{y_r}(v\sin\xi - v_2)$$
$$+ \lambda_{v_r}(T-D)/m + \mathcal{L} = 0 \tag{53}$$

In this axis system

$$\lambda_{x_r}^o = -\mathcal{L}^o(t_f)/(v(t_f) - v_1) \tag{54}$$

and

$$\lambda_{v_r}^o = -m/(T-D)\cdot[\lambda_{x_r}^o(v - v_1) + \ell] \tag{55}$$

Boundary Layer Solution (Fast Problem)

In the outer solution it was assumed that the missile can change its velocity instantaneously towards the direction $\vec{v}_d$. The turning transients which are ignored in the outer solution are accounted for in this boundary layer. We model the missile turn dynamics in going from the current missile SAA heading to the outer solution optimal SAA heading ξ^o. This analysis results in a nonlinear solution for optimal lift $(\tilde{L})$ in the $(v - v_d)$ plane. It should be noted that as ξ goes to ξ^o, L must tend to zero, or the first boundary layer solution must asymptotically approach the outer solution as the turn is completed.

The fast problem is solved by using the time-stretching transformation $\tau = t/\epsilon$ and letting $\epsilon \to 0$. The Hamiltonian for the boundary layer problem is

$$H^1 = \lambda_{x_r}^1(v\cos\xi - v_1) + \lambda_{y_r}^1(v\sin\xi - v_2) + \lambda_{v_r}^1(T-D)/m$$
$$+ s_\xi^1 v_o^2\tilde{L}/vdL_{\max} + \mathcal{L} = 0 \tag{56}$$

The optimization of this Hamiltonian leads to

$$\frac{\partial H^1}{\partial\tilde{L}} = \frac{\partial\mathcal{L}}{\partial\tilde{L}} - \frac{\lambda_{v_r}^o}{m}\cdot\frac{2K\tilde{L}}{qs} + \frac{s_\xi^1 v_o^2}{vdL_{\max}} = 0 \tag{57}$$

The first boundary-layer Hamiltonian H^1 can be expressed in terms of the outer solution Hamiltonian H^o as

$$H^1 = H^o(\lambda_{xr}^o, \lambda_{yr}^o, \lambda_{vr}^o, \xi) - \lambda_{vr}^o/m\cdot K\tilde{L}^2/qs$$
$$+ \lambda_\xi^1\cdot v_o^2\tilde{L}/vdL_{\max} + \mathcal{L} - \mathcal{L}^o = 0 \tag{58}$$

Eliminating λ_ξ^1 from Eqs. (50) and (51),

$$H^o + \frac{\lambda_{vr}^o}{m}\frac{K\tilde{L}^2}{qs} + \mathcal{L} - \mathcal{L}^o - \tilde{L}\frac{\partial\mathcal{L}}{\partial\tilde{L}} = 0 \tag{59}$$

As $\xi \to \xi^o = 0$, $H^o(\lambda_{xr}^o, \lambda_{yr}^o, \lambda_{vr}^o, \xi) \to 0$ and $\tilde{L} \to \tilde{L}^o = 0$. The control $\tilde{L}$ in the boundary layer merges asymptotically into the control in the outer solution. The necessary conditions used in computing the optimal control $\tilde{L}$ are the same as in the reduced problem except that the adjoints $\lambda_{x_r}^o$, $\lambda_{y_r}^o$, $\lambda_{v_r}^o$ are known accurately only to zero order.

Table 3 Singular perturbation control laws

Case	$\mathcal{L}$	Guidance law
1) Minimum time	1.0	$\tilde{L}=\sqrt{\dfrac{qs}{K(-\lambda_{vr}^{o}/m)}}\cdot H^{o}$
2) Minimum drag	$qsC_{do}+\dfrac{KL^2}{qs}$	$\tilde{L}=\sqrt{\dfrac{qs}{K(1-\lambda_{vr}^{o}/m)}}\cdot H^{o}$
3) Combination of min time and min drag $0\leq p\leq 1$	$pD+(1-p)$	$\tilde{L}=\sqrt{\dfrac{qs}{K(p-\lambda_{vr}^{o}/m)}}\cdot H^{o}$
4) General function of lift	$f(L)$	$\dfrac{\lambda_{vr}}{m}\cdot\dfrac{K\tilde{L}^2}{qs}-\tilde{L}\dfrac{\partial f(\tilde{L})}{\partial\tilde{L}}+[f(\tilde{L})-f(\tilde{L}^{o})]+H^{o}(\xi)=0$

Choice of $\mathcal{L}$

One of many reasonable cost functions $\mathcal{L}$ may be selected depending on the missile and sensor characteristics and other requirements. Some of the possible choices are:

1) Minimum time

$$\mathcal{L}=1 \tag{60}$$

This choice is useful in short-range missiles, particularly in dogfights.

2) Minimum total drag

$$\mathcal{L}=D \tag{61}$$

This form may be useful in long flights when maximum energy should be saved for unexpected target maneuvers in the final phase, and when the missile thrust may be modulated.

3) Missile stability

$$\mathcal{L}=f(L^2) \tag{62}$$

The function f may be selected in many ways to ensure that excessive lift is not commanded such that the flight condition is stable. This form may also be used to optimize control surface deflections or minimize battery power requirements.

4) Combinations of the above

The current desired value of the missile lift is determined for each of these cost functions using Eq. (59). These control laws are shown in Table 3.

Implementation

Figure 3 shows the major subsystems of the missile. The SP guidance law provides acceleration commands to the missile autopilot (see Table 3 for explicit expressions). Thrust was not optimized and was determined by the motor characteristics. However, the algorithm can be easily modified to include thrust magnitude control. The example in the next section uses minimum time SP guidance law (Table 3, case 1). In contrast, the acceleration commands generated by PN guidance are given by

$$a_m=K_{\text{nav}}\cdot v_c\dot{\sigma} \tag{63}$$

Range and range rate measurements and some estimate of the target motion are needed to mechanize the SP guidance law. Range and rate measurements can be obtained by using an active radar. One version of PN guidance [Eq. (63)] uses both LOS rate measurements and closing rate information. This requires the use of an active seeker; in the absence of an active seeker, closing velocity is replaced by missile velocity and the navigation gain must be adjusted for each engagement to improve performance.

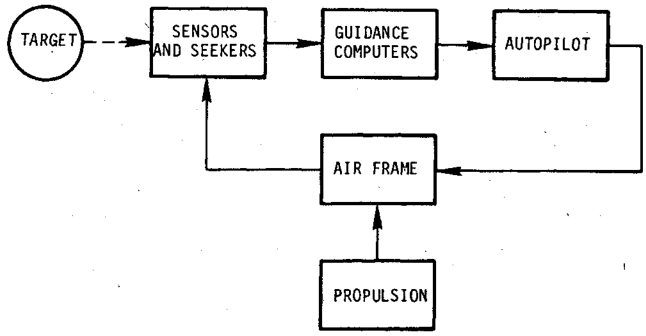

Fig. 3 Integration of guidance law with missile subsystems.

In SP guidance, the range and range rate information is converted to estimates of time-to-go and optimal heading. This computation is a large percentage of the total computation involved at each guidance step. This can be done very efficiently by using Newton-Raphson's method and the previous value of time-to-go as the initial guess. These computations can be done in one of the currently available microprocessors, such as the Intel 8085.

Guidance signals are affected by sensor noise due to radar jamming devices, scale factor errors, linearity errors, signal processing errors, and random errors. In general, range and look angle measurements are less susceptible to errors than range rate measurements. The computation of time-to-go involves only range measurements. All these errors are compensated to a certain extent by the feedback nature of the algorithm. Preliminary analysis and simulation have shown that the algorithm is not very sensitive to estimates of target velocity.

PN guidance may have an advantage over SP guidance if the target maneuvers randomly. However, a "smart" target should, assuming that the missile has significant turn rate and velocity advantage over the target, force the missile to 1) command more than the maximum turn rate, 2) exceed the slew rate capability of the seeker, and/or 3) stress the dynamical reference of the airframe/autopilot. The "smart" target evasive algorithm[9] is designed to bring out these characteristics in an air-to-air engagement. The SP guidance has shown significant improvement over PN guidance against a "smart" target.

Table 4 Summary of missile-target characteristics used in guidance law computation

Missile	
Initial velocity, ft/s	1000
Initial weight, lb	165
Burn-out weight, lb	115
Burn-out time, s	2.6
Boost thrust, lb	4700
s, ft^2	1.0
C_{do}	0.3
K	0.0025
Maximum acceleration	Function of flight condition
Target	
Initial velocity, ft/s	1000
Acceleration, g	8

Table 5 Performance of guidance laws

Range	2000 ft PN	2000 ft SP	5000 ft PN	5000 ft SP
Miss distance, ft	12.6	2.55	0.13	0.07
Time, s	0.872	0.872	1.872	1.872
Maximum acceleration, g	75.0	68.0	60.0	30.0
Final velocity, ft/s	1554	1554	2536	2536
Final weight, lb	148.3	148.3	129.1	129.1

Example

The example presented in this section demonstrates the characteristics of the technique and compares it to the conventional pro nav guidance law. Both guidance laws were tested on a six-degree-of-freedom (6-DOF) simulation.[10]

A bank-to-turn short-range air-to-air missile is fired against an aircraft-type maneuvering target. Important missile and target parameters are summarized in Table 4. Three engagement scenarios are considered: 1) tail chase, 2) initial velocities at right angles, and 3) head-on approach. In each case, at a certain distance the target makes an 8-g turn (equal acceleration in horizontal and vertical planes) for various target-to-missile ranges. There is a guidance initiation delay of 0.4 in each engagement.

In tail chase and orthogonal approach scenarios, the pro nav performs reasonably well. The SP solution is significantly better, nevertheless. The most dramatic improvements are obtained in the most difficult head-on approach.

Table 5 shows a comparison of the final conditions for pro nav and SP guidance laws for 2000 ft and 5000 ft initial ranges. The PN guidance does not intercept the target for an initial range of 2000 ft.

Figures 4 and 5a-5d provide further comparison of the guidance laws. Figure 4 shows the missile-target trajectories for PN and SP guidance obtained from the 6-DOF simulation. The commanded and missile acceleration in the pitch plane are shown in Figs. 5a and 5b. The missile maneuver consists of quick roll so that the line of sight rate vector is located in the pitch plane of the missile. The roll rate command is limited to ± 500 deg/s to avoid roll rate gyro limits and acutator limits. Roll angle time-history is shown in Fig. 5c. Figure 5d shows that angle of attack time history for both guidance laws.

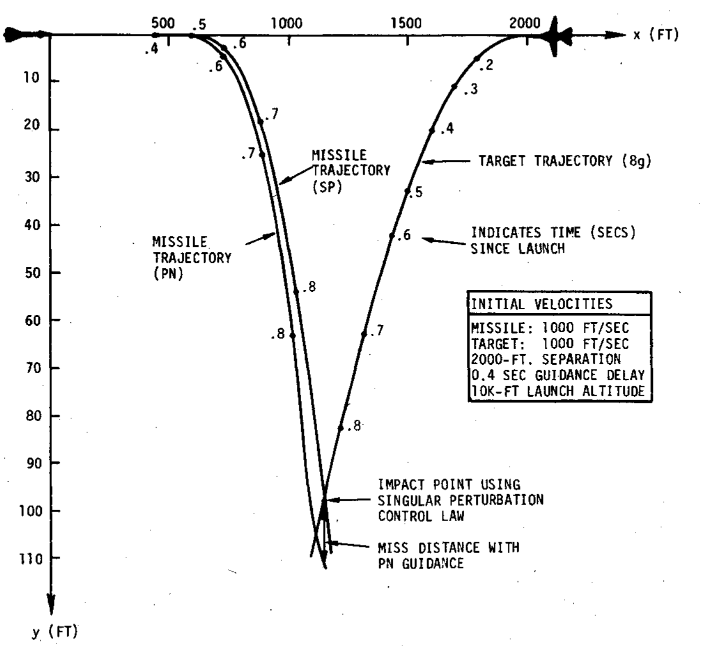

Fig. 4 Missile-target trajectories for PN and SP guidance.

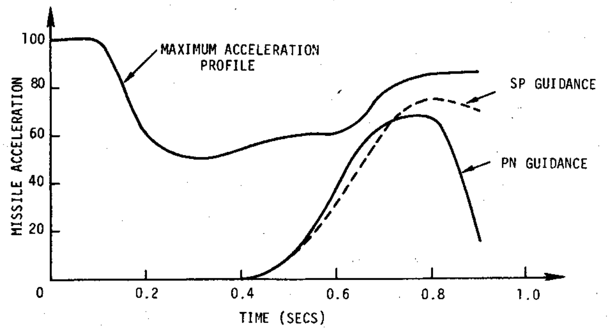

Fig. 5a Missile acceleration in the pitch plane for PN and SP guidance.

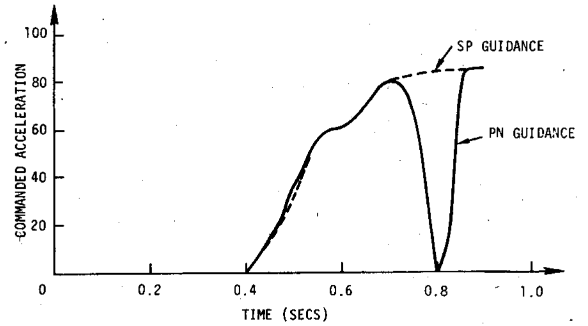

Fig. 5b Commanded acceleration in the pitch plane.

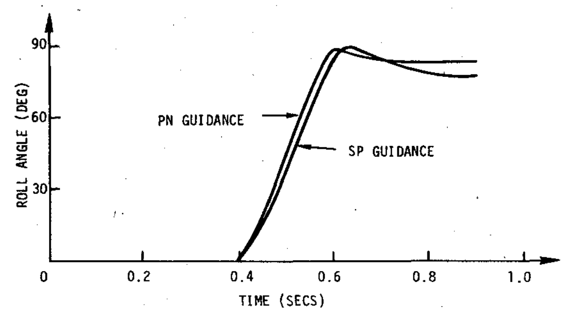

Fig. 5c Roll angle time history.

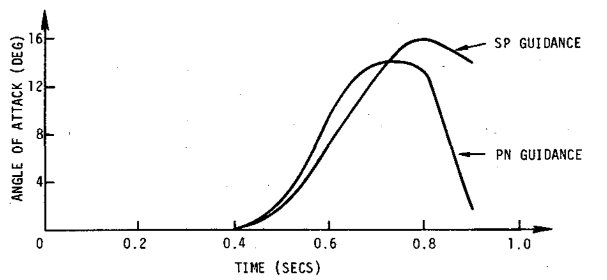

Fig. 5d Angle-of-attack time history.

The lift for the SP guidance law increases towards the latter part. This is primarily because the approximations in the SP method become poorer when the time-to-go is very small. Extensions to the SP method are being worked out to apply first order corrections to the slow solution to further improve the performance.

Conclusions

The singular perturbation (SP) technique developed in this paper for the guidance of short-range air-to-air missiles provides a significnt improvement over pro nav guidance, particularly in difficult engagements. The SP guidance is also simple enough such that real-time implementation should be straightforwård even using current microprocessors. The guidance law does not require the solution of any two-point boundary value problems. The three major components of the intercept solution are: 1) determination of the desired missile flight path for intercept, 2) computation of the plane of missile maneuver, and 3) the commanded missile acceleration in the maneuver plane. New solution methods are developed for each of these three steps, leading to a simple and effective guidance law. The computations are further simplified by the introduction of an artificial target.

The guidance law is based on an estimate of target acceleration. Its performance will, to some extent, depend on how well the target acceleration may be estimated. If the target acceleration cannot be determined, its estimate must be set to zero. Preliminary investigations have shown that sensitivity to errors in estimates of target acceleration is small. Therefore, it is not necessary to know accurately the future behavior of the target in the computation of the guidance commands.

The performance of the derived guidance law also depends upon the validity of the approximations used in singular perturbation solutions. The solution deviates from the optimal as time-to-go decreases. Significant differences are noted when the time to go is in the same range as the autopilot time constant. First order corrections can be applied to the outer solution (slow solution) to provide a better approximation. Such corrections are currently being developed.

Acknowledgments

The authors wish to thank T. Riggs and G. Gonzales of Eglin Air Force Base and their colleagues at Systems Control, Inc. (Vt) for their useful discussions on the subject of this paper. The work is supported by Eglin Air Force Base under Contract F08635-78-C-0206.

References

[1] Pastrick, H.L., Seltzer, S.M., and Warren, M.E., "Guidance Laws for the Short-Range Tactical Missiles," AIAA Paper 79-0059, 17th Aerospace Sciences Meeting, New Orleans, La., Jan. 1979.

[2] Murtaugh, S.A. and Criel, H.E., "Fundamentals of Proportional Navigation," *IEEE Spectrum*, Vol. 3, Dec. 1966, pp. 75-78.

[3] Arbenz, K., "Proportional Navigation of Nonstationary Targets," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 6, July 1970, pp. 455-457.

[4] Axelband, E.I. and Hardy, F.W., "Quasi-Optimum Proportional Navigation," *IEEE Transactions on Automatic Control*, Vol. 15, Dec. 1970, pp. 620-627.

[5] Guelman, M., "The Closed-Form Solution of True Proportional Navigation," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 12, July 1976, pp. 472-482.

[6] Bryson, A.E. and Ho, Y.C., *Applied Optimal Control*, Xerox Blaisdell, Waltham, Mass., 1969.

[7] Kokotovic, P.V., O'Malley, R.E., Jr., and Sannuti, P., "Singular Perturbations and Order Reductions in Control Theory," *Automatica*, Vol. 12, 1976, pp. 123-132.

[8] Sridhar, B. and Gupta, N.K., "Missile Guidance Laws Based on Singular Perturbation Methodology," SCI Tech. Memo 5270-05, Nov. 1978.

[9] Riggs, T.L., "Simplified 'Smart' Target Evasive Algorithm," Eglin AFB Memo DLMA-TM-78-3, June 1978.

[10] Emmert, R.I., "Bank-to-Turn Steering for Tactical Missiles," Rockwell report to Air Force Armament Laboratory (DLMA), Eglin, Fla., 1977